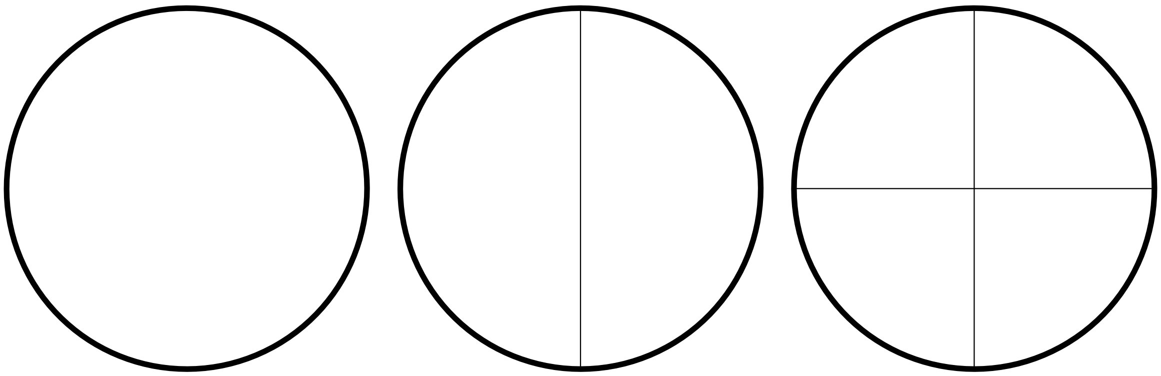
T-Shirt Explanation: Westmont's 29th Annual Mathematics Contest

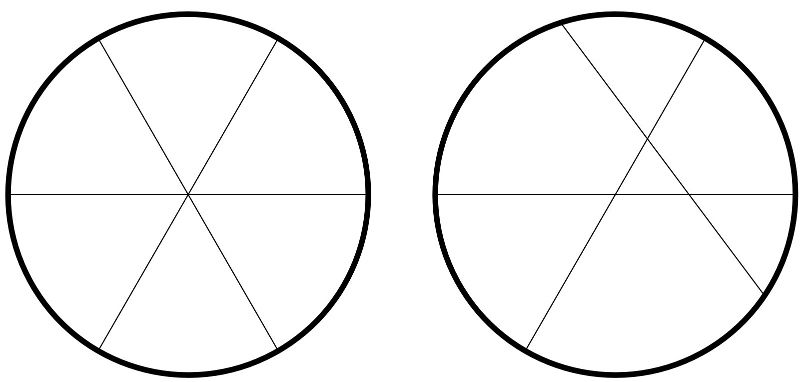
How many pieces of pizza can you get with N cuts? When you make no cuts ($N = 0$) you obviously get one piece—the whole pizza! With one cut ($N = 1$) or two cuts ($N = 2$) you can get a maximum of 2 or 4 or pieces, respectively. Figure 1 illustrates these observations.



(a) $N = 0$; Max. Pieces = 1 (b) $N = 1$; Max. Pieces = 2 (c) $N = 2$; Max. Pieces = 4

Figure 1: Maximum number of pieces with 0, 1, or 2 cuts

So far we've obtained the maximum number of pieces by cutting "down the middle," but this would be a bad strategy in general. Doing so with $N = 3$ cuts results in six pieces, but as Figure 2 shows you can do better by making the third cut cross each of the previous cuts (as opposed to going through the center of the pizza).



(a) $N = 3$; Max. Pieces $\neq 6$ (b) $N = 3$; Max. Pieces = 7

Figure 2: Maximum number of pieces with 3 cuts

Getting a formula for the maximum number of pieces involves an application of mathematical recursion. To this end, let $f(N)$ be the maximum number of pieces that can be obtained with N cuts. As we've already observed, $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, and $f(3) = 7$. Figure 2 illustrated that, to maximize the number of pieces, a new cut must cross each of the previous cuts.

Imagine, then, that we have the maximum number of pieces obtainable with $N - 1$ cuts. By definition, this maximum number equals $f(N - 1)$. To get the maximum number of pieces with the Nth cut, we must make sure it crosses each of the $N - 1$ previous cuts. Each time it intersects a previous cut it divides an existing piece of pizza in two, thus forming $N - 1$ new pieces by the time it hits the last cut.

As you can see from Figure 3, however, one additional piece will be formed when the Nth cut reaches the boundary of the pizza.

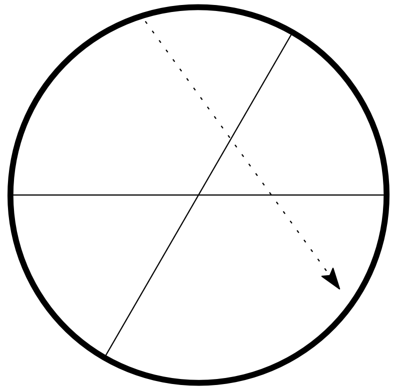


Figure 3: Forming a new piece at the boundary

We see, then, that the Nth cut adds a total of N new pieces to the previously existing $f(N-1)$ pieces of pizza. In other words,

$$f(N) = f(N-1) + N. \tag{1}$$

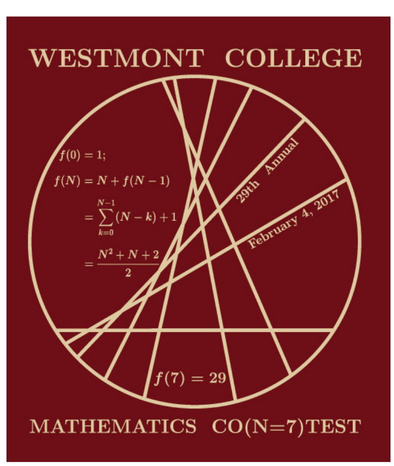
Keep in mind that this relation holds for *all* values of $N \geq 1$, so, for example, letting $N-1$ take the role of N in Equation (1) gives $f(N-1) = f(N-2) + N - 1$. This observation, combined with the fact that $f(0) = 1$, enables Equation (1) to be solved with a technique known as *unfolding*:

$$\begin{aligned} f(N) &= f(N-1) + N \\ &= \big(f(N-2) + N - 1\big) + N \\ &= \Big(\big(f(N-3) + N - 2\big) + N - 1\Big) + N \\ &\ \vdots \\ &= f(0) + 1 + 2 + 3 + \cdots + (N-2) + (N-1) + N. \\ &= 1 + 1 + 2 + 3 + \cdots + (N-2) + (N-1) + N. \end{aligned}$$

A standard result supposedly discovered by the great German mathematician Carl Friedrich Gauss when he was in primary school is that $1+2+3+\cdots+(N-2)+(N-1)+N = \frac{N(N+1)}{2}$. Therefore,

$$f(N) = 1 + \frac{N(N+1)}{2} = \frac{N^2+N+2}{2}.$$

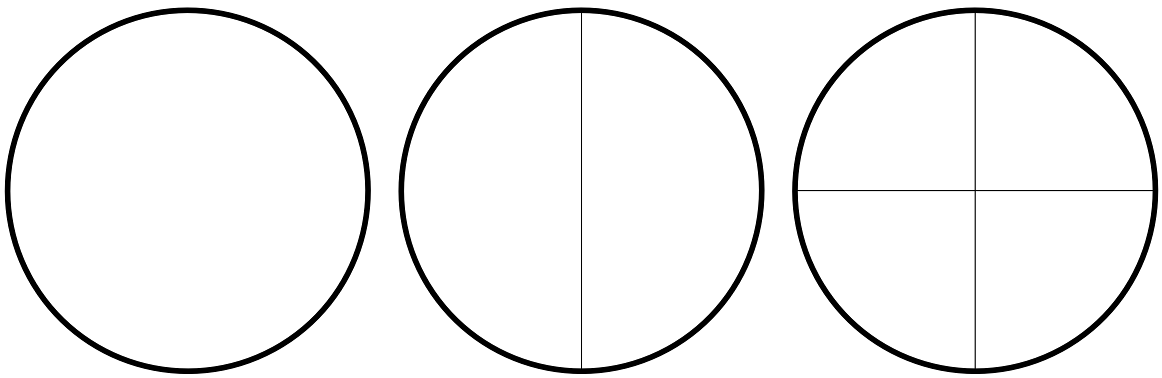
With $N = 7$ we can thus get $f(7) = \frac{7^2+7+2}{2} = 29$ pieces, which the T-shirt, shown below, illustrates.



Food for Thought

It is doubtful that anyone would want to be served the smallest piece of the pizza shown on the previous page! It was carved up that way so as to allow room for a proper display of the recurrence relation as it unfolded. How should a pizza be carved up so as to ensure that the piece with the smallest area is as large as possible? Is there a formula for that area?

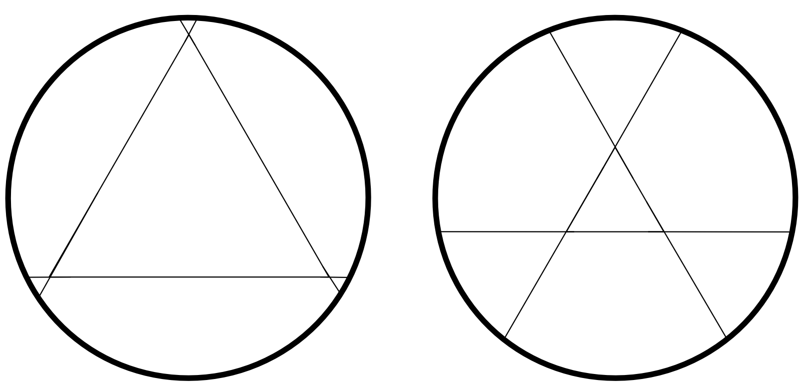
Figure 4 illustrates that, for $N = 0$, $N = 1$, and $N = 2$, the maximum area of the smallest piece is what we would like: the areas of each of the pieces are identical. In these cases the fraction of area occupied by each piece of a pizza is $\frac{1}{f(N)} = \frac{1}{\text{Maximum Number of Pieces}}$.



(a) $N = 0$; Area $= \frac{1}{f(0)} = 1$ (b) $N = 1$; Area $= \frac{1}{f(1)} = \frac{1}{2}$ (c) $N = 2$; Area $= \frac{1}{f(2)} = \frac{1}{4}$

Figure 4: Each piece has equal area

What happens with three cuts? Can a mechanism be devised so that each piece has a fractional area equal to $\frac{1}{f(3)} = \frac{1}{7}$? The strategy depicted in Figure 5 will not work. The central equilateral triangle of Figure 5a can be contracted continuously so that, at some point before it shrinks to the one in Figure 5b, its area will equal that of the three outer triangular sectors. But at this point the remaining three quadrilateral sectors will all have a larger area than the triangular sections.



(a) Initial 3-cut (b) Final 3-cut

Figure 5: Attempting to optimize the minimal area

To date, there is no known formula for the maximum area of the smallest piece, given that a pizza is cut N times. The curious reader is invited to discover one!